The Simple Mathematics of Optimal Auctions

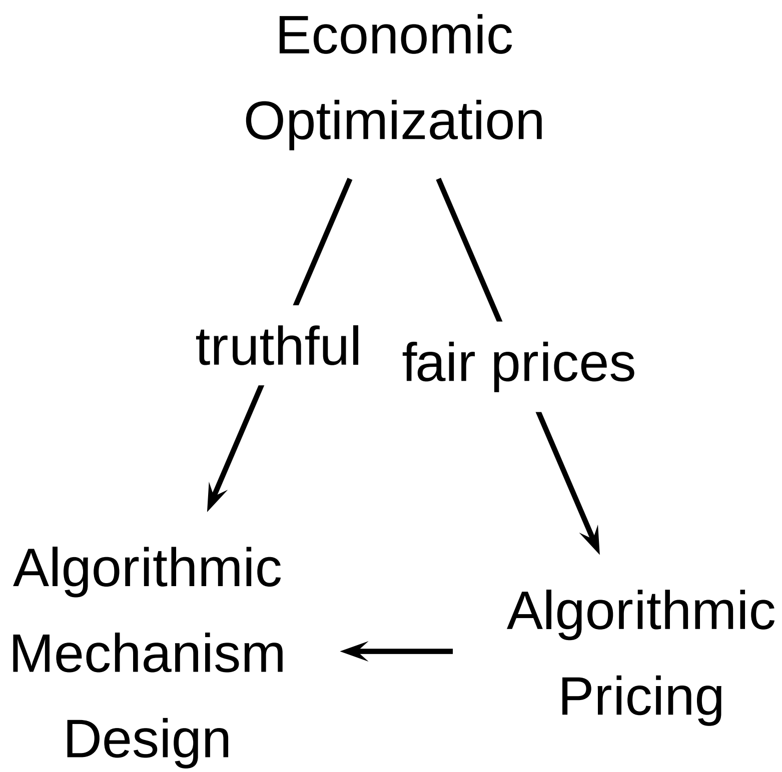
Jason D. Hartline

(joint with Maria-Florina Balcan,

Nikhil Devanur, and Kunal Talwar)

March 28, 2007

Economic Optimization



Overview

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   (a) Algorithmic pricing.
   (b) Mechanism design via pricing.
2. Generalize to limited supply setting:
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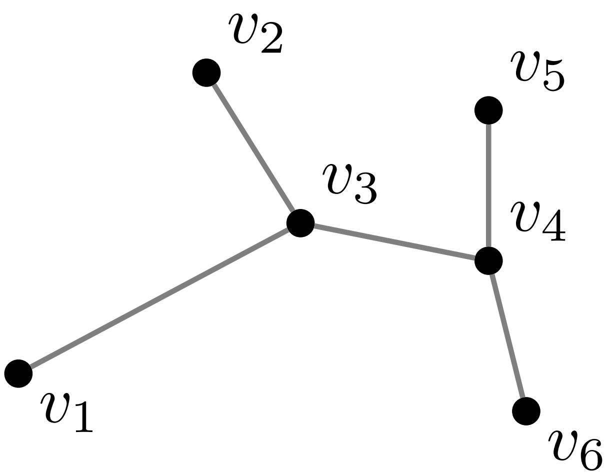
3. Generality & conclusions.

Example: Path Pricing

Example: *Edge pricing selling paths.*



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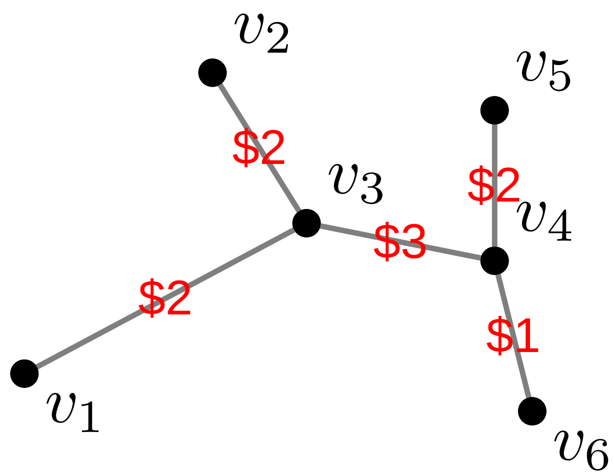

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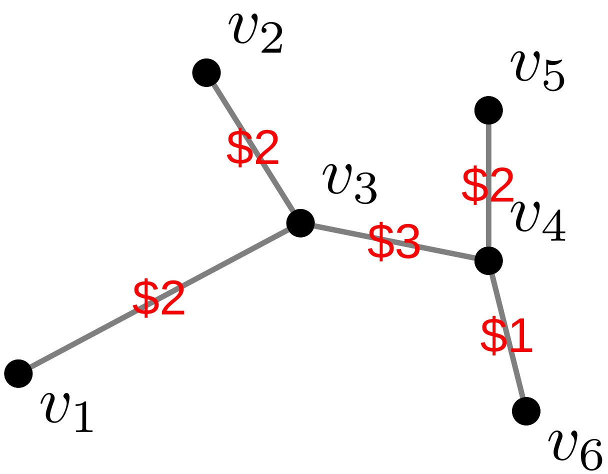

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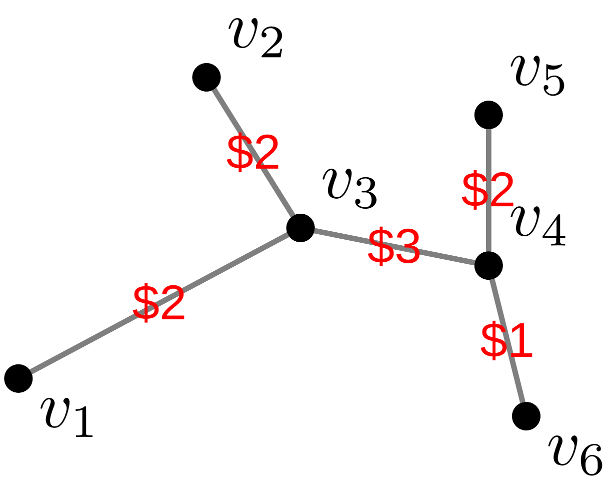

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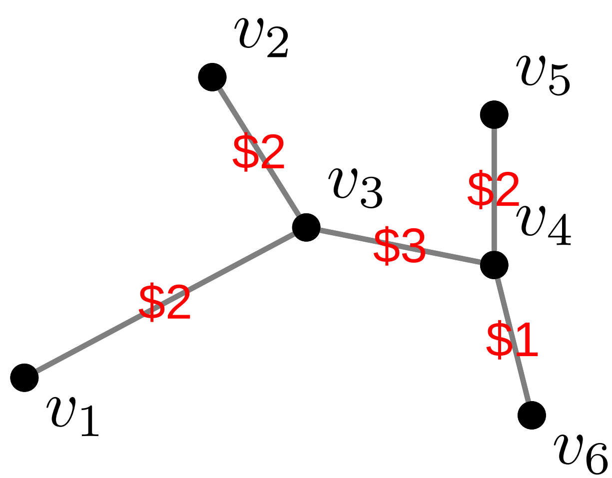

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Goal: price edges to maximize objective.

Unlimited Supply Algorithmic Pricing

The *Unlimited Supply Algorithmic Pricing* problem:

Given:

- unlimited supply of stuff.
- Set S of n consumers and their preferences for stuff.
- class $\mathcal{G}$ of reasonable offers.

Design: Algorithm to compute optimal offer from $\mathcal{G}$.

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How can we compute $\mathrm{opt}_{\mathcal{G}}$?

1. Sort valuations: $v_1 \geq \ldots \geq v_n$
2. Output v_i to maximize $i \times v_i$.

Literature

Algorithmic Pricing in the Literature

- unlimited supply (mostly).
- many interesting special cases.
- includes work of: Gagan Aggarwal, Maria-Florina Balcan, Avrim Blum, Patrick Briest, Shuchi Chawla, Eric Demaine, Tomás Feder, Uri Feige, Venkat Gurusuami, MohammadTaghi Hajiaghayi, Anna Karlin, David Kempe, Vladlin Koltun, Robert Kleinberg, Piotr Krysta, Clare Mathieu, Frank McSherry, Rajeev Motwani, and An Zhu.

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- hard (even to approximate).

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Auction Problem

The *Unlimited Supply Auction Problem*:

Given:

- unlimited supply of stuff.
- Set S of n bidders with preferences for stuff.
- class $\mathcal{G}$ of reasonable offers.

Design: Single round, sealed bid, *truthful* auction with profit near that of $\mathrm{OPT}_{\mathcal{G}}$.

Recall Notation:

- $g(i) =$ payoff from bidder i when offered g.
- $g(S) = \sum_{i \in S} g(i)$.
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Generalization of auction from [Goldberg, Hartline, Wright '01]:

Random Sampling Optimal Offer Auction, RSOO$_{\mathcal{G}}$

1. Randomly partition bidders into two sets: S_1 and S_2.
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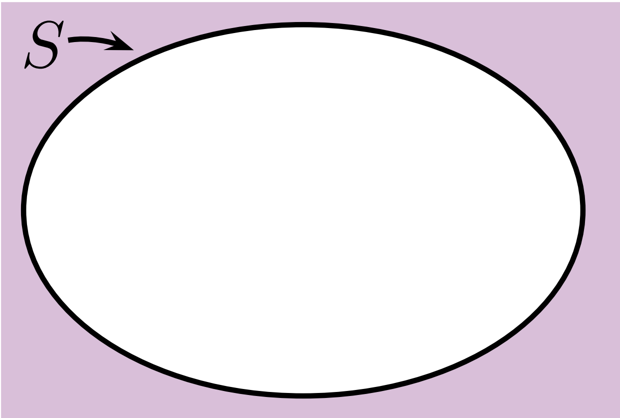

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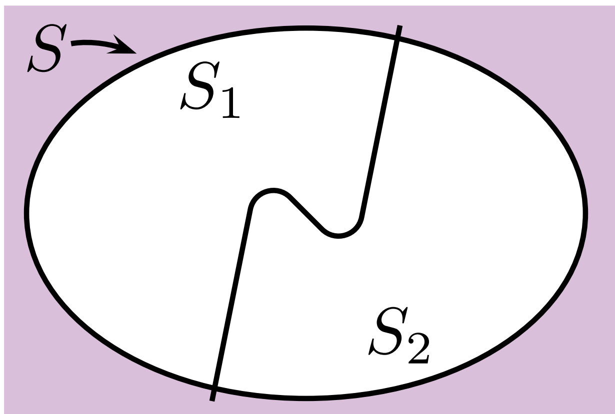

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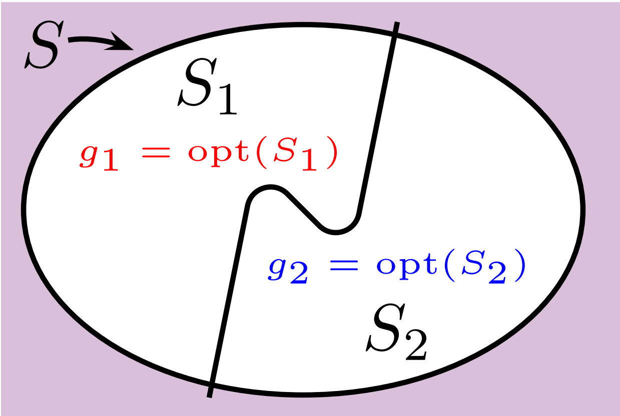

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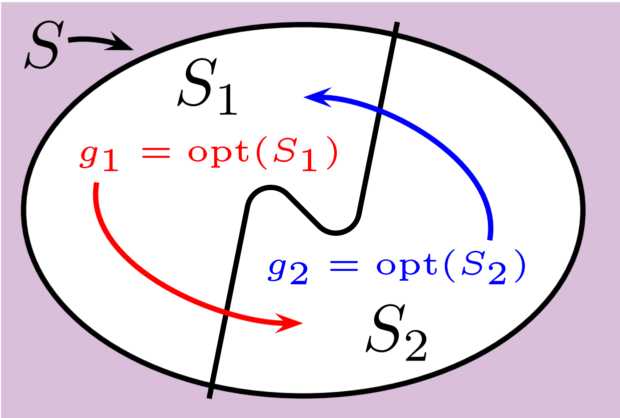

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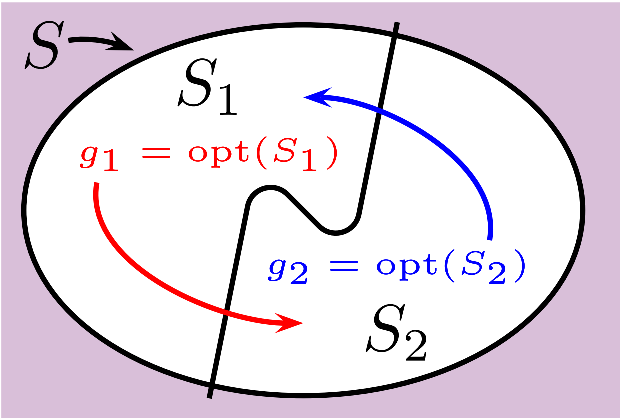

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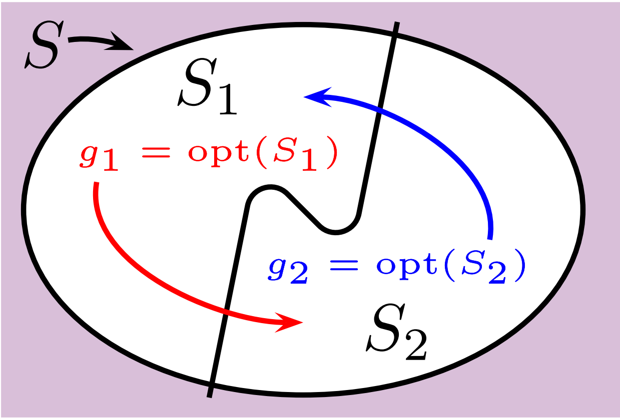

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Question: when does RSOO$_{\mathcal{G}}$ perform well?

Performance Analysis

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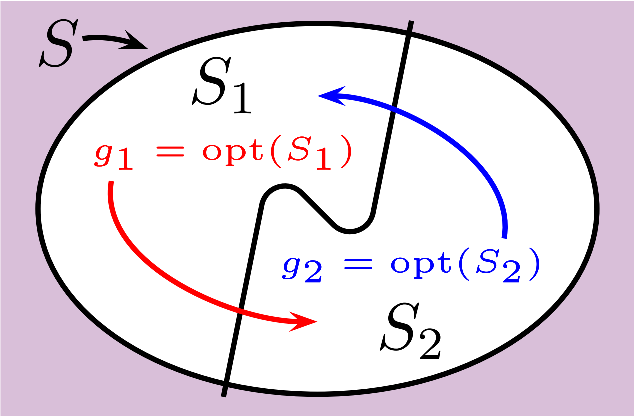

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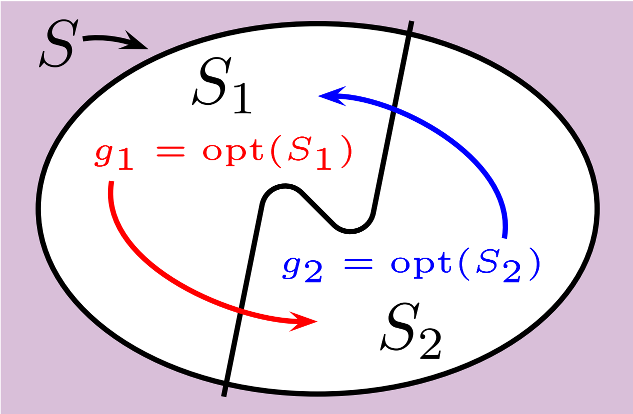

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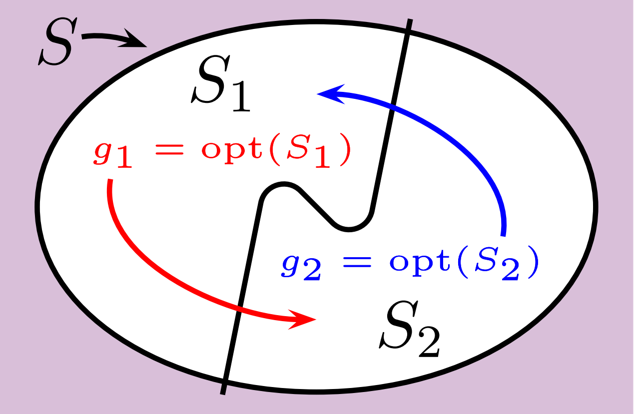

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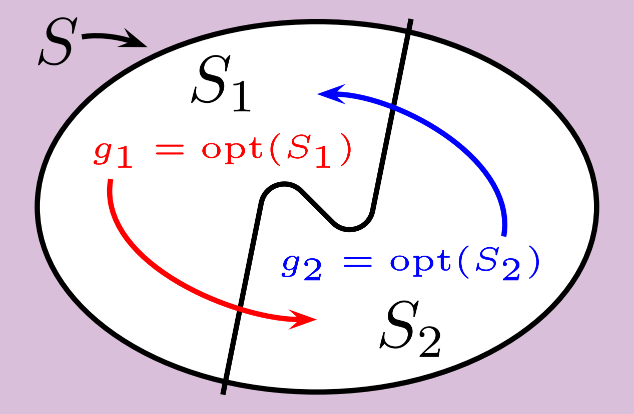

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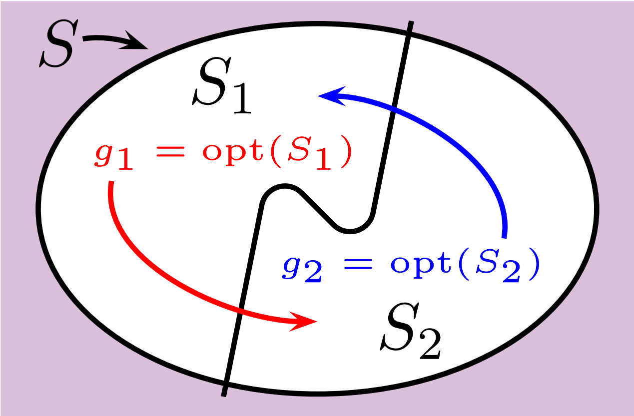

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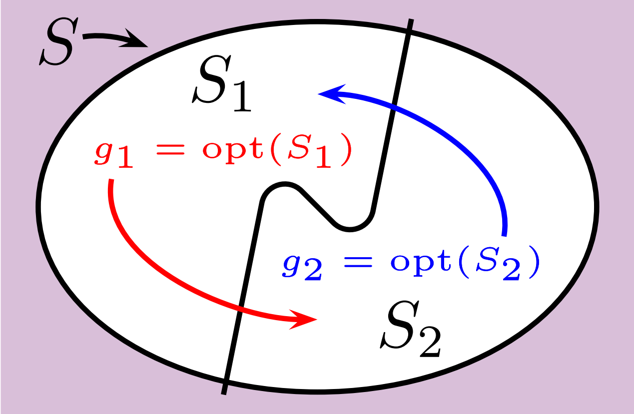

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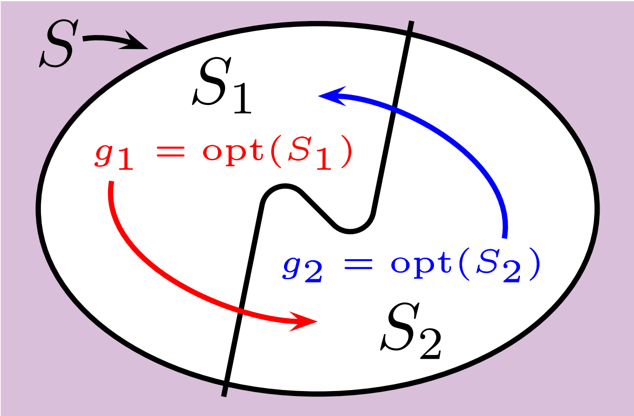

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Interpretation: convergence rate is $O(h \log |\mathcal{G}|)$.

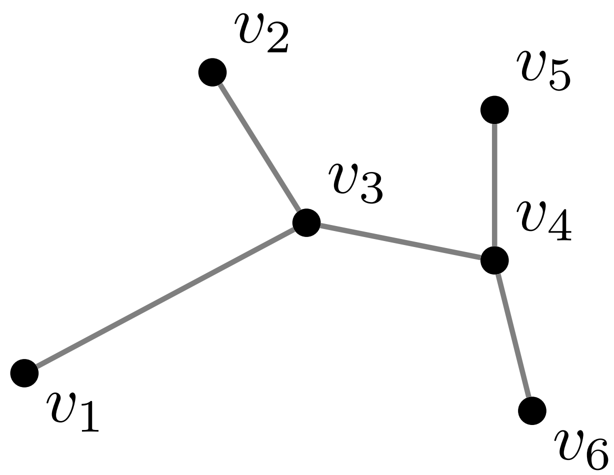
Example: Digital Good Auctions

Example: Digital good with discretized prices.

- Bidders with valuations in $[1, h]$ for a good.
- Reasonable offers: $\mathcal{G} = \{\text{price } 2^i$ for $i \in \{1, \ldots, \log h\}\}$.
- Convergence Rate: $O(h \log |\mathcal{G}|) = O(h \log \log h)$

Example: Path Auctions

E.g., selling bandwidth on paths in a graph.



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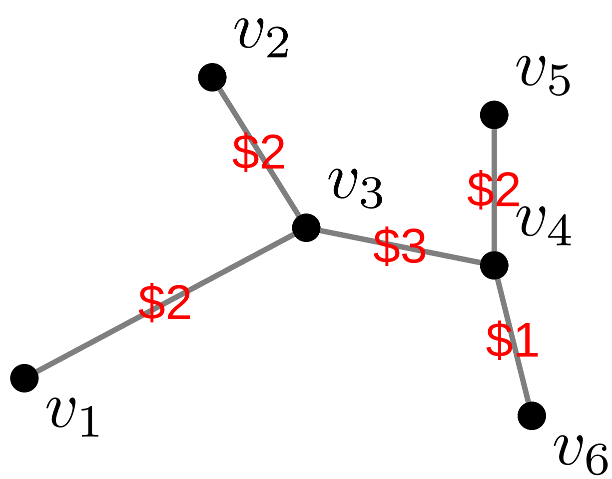

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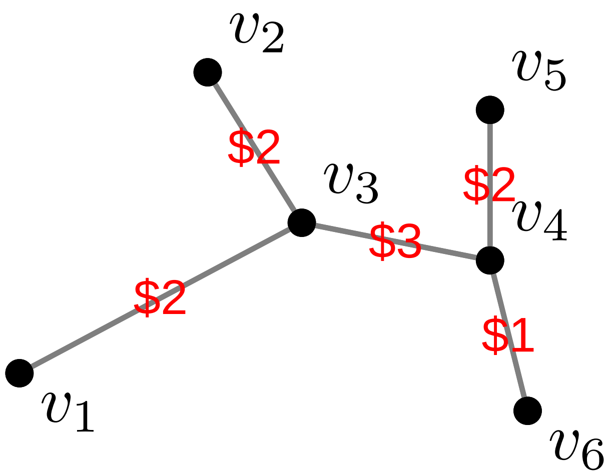

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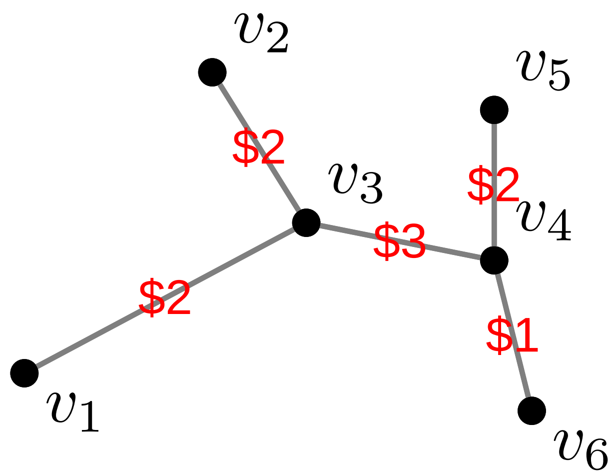

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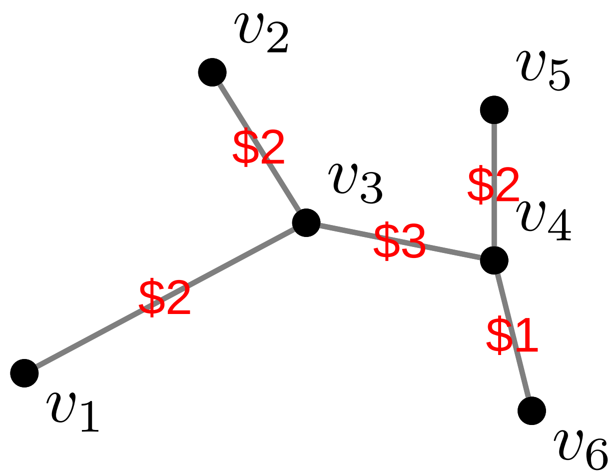

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Result: Convergence rate of RSOO$_{\mathcal{G}}$ is $O(hm \log\log h)$.

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Limited Supply Algorithmic Pricing

The *Limited Supply Algorithmic Pricing* problem:

Given:

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What if $x_j(S, g) > C_j$?

Dealing with Excess Demand

Two approaches:

- restrict algorithm. [Gurusuami et al. '05]
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  What is the payoff of offer g?

Single Commodity and Uniform Knapsack

A knapsack problem:

- consumer payoffs: $p(1, g), \ldots, p(n, g)$.
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Theorem: When $x(i,S) > C$ then

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Proof: via reduction to uniform payoff case (i.e., $p(i,g) = 1$)

Definition: *Estimated payoff* of g on S: $P(S,g,C) = \frac{C \cdot p(S,g)}{\max\{C, x(S,g)\}}$

Limited Supply Algorithmic Pricing

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- $\mathrm{opt}_{\mathcal{G}}(S,C) = \mathrm{argmax}_{g \in \mathcal{G}}\, P(S,g,C)$.
- $\mathrm{OPT}_{\mathcal{G}}(S,C) = \max_{g \in \mathcal{G}}\, P(S,g,C)$.

Algorithmic Pricing Goal: compute $\mathrm{opt}_{\mathcal{G}}(S,C)$.

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Limited Supply Random Sampling Auction

Generalization of auction from [Borgs et al. '05]:

Random Sampling Limited Supply Auction, RSLS$_{\mathcal{G}}$

1. Randomly partition bidders into two sets: S_1 and S_2.
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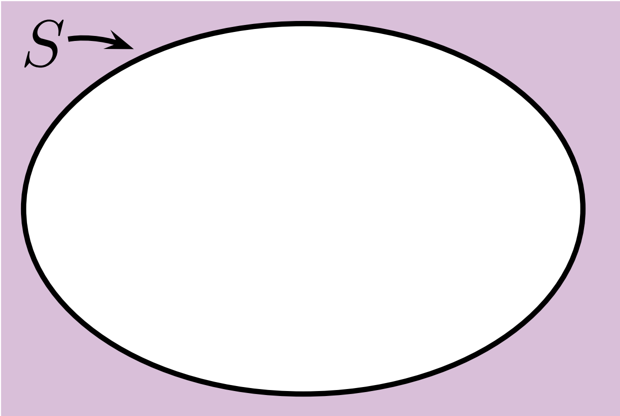
3. Offer g_1 to S_2 and g_2 to S_1.



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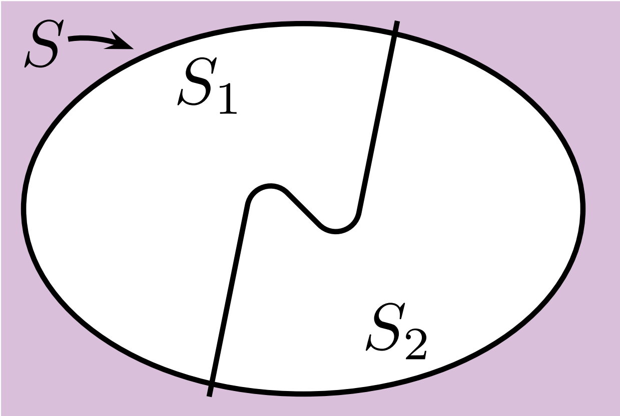

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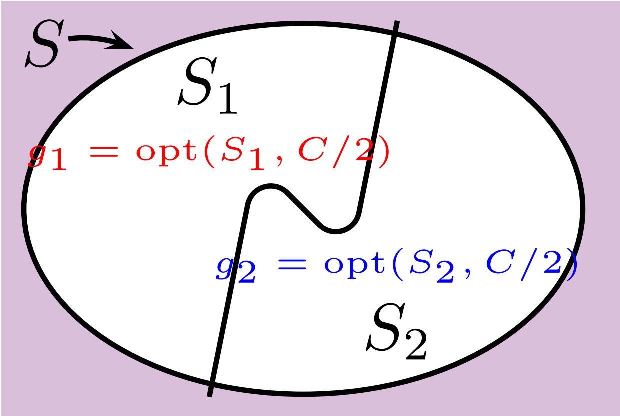

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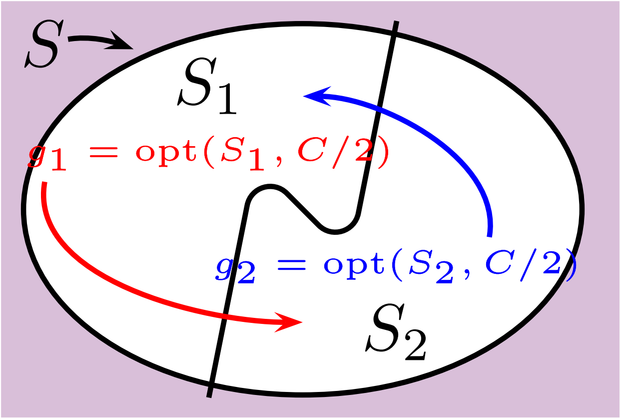

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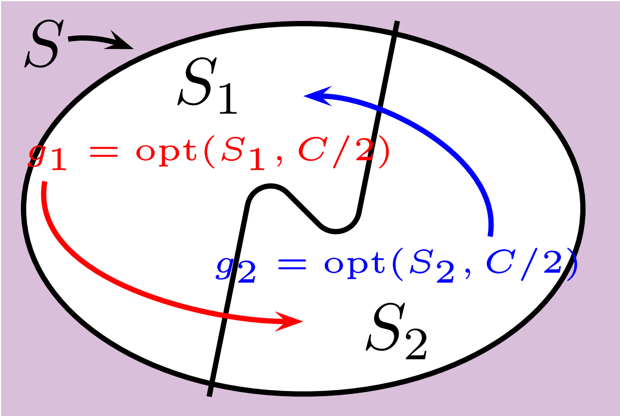

Fact: RSLS$_{\mathcal{G}}$ is truthful.

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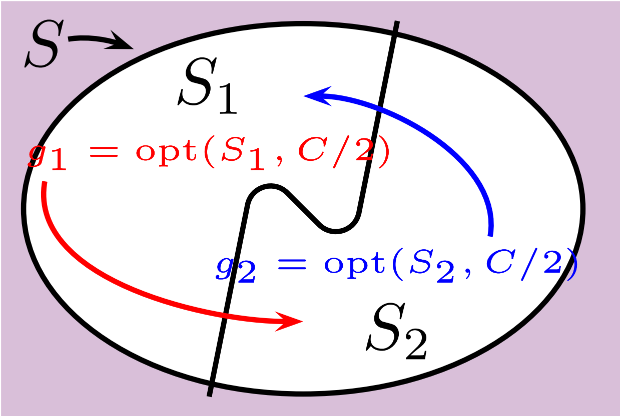

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Question: when does RSLS$_\mathcal{G}$ perform well?

RSLS$_{\mathcal{G}}$ Performance

Theorem: With probability $1 - \delta$,

$$\text{Profit} \geq (1 - \epsilon)\,\mathrm{OPT}_{\mathcal{G}}$$

when $\frac{\mathrm{OPT}_{\mathcal{G}}}{p_{\max}}$ and $\frac{C}{x_{\max}}$ are $O(\frac{1}{\epsilon^2}\log\frac{4|\mathcal{G}|}{\delta})$.

Proof Sketch:

1. With probability $1 - \delta$ all g are *ϵ-good*.
   (with respect to $p(S, g)$ and $x(S, g)$).
2. Thus, g_1 and g_2 are ϵ-good.
3. $P(S_1, g_2, C/2) \geq (1 - \epsilon')P(S_2, g_2, C/2)$.
4. $P(S_1, g^*, C/2) + P(S_1, g^*, C/2) \geq (1 - \epsilon'')P(S, g^*, C)$.
5. Profit $\geq (1 - \epsilon''')\,\mathrm{OPT}_{\mathcal{G}}(S, C)$.

Example Analysis:

Claim: $P(S_1, g_2, C/2) \geq (1 - \epsilon')P(S_2, g_2, C/2)$.

Sketch:

$$
\begin{aligned}
P(S_1, g_2, C/2) &= \tfrac{C}{2}\frac{p(S_1, g_2)}{\max\{C/2, x(S_1, g_2)\}}) \\
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Key Fact for Theorem: $p(S, g)$ and $x(S, g)$ are sums of i.i.d. variables.

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Economic Optimization

Economic

Optimization

truthful

fair prices

Algorithmic

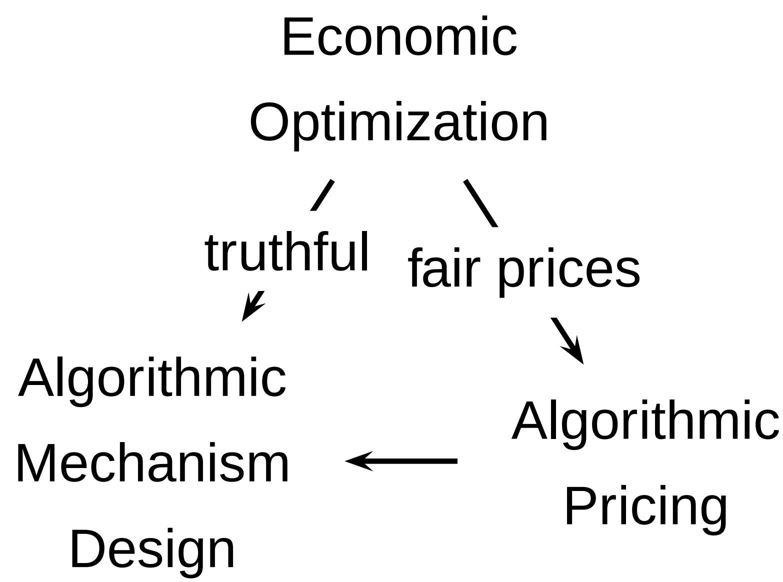
Mechanism

Design

Algorithmic

Pricing

Economic Optimization



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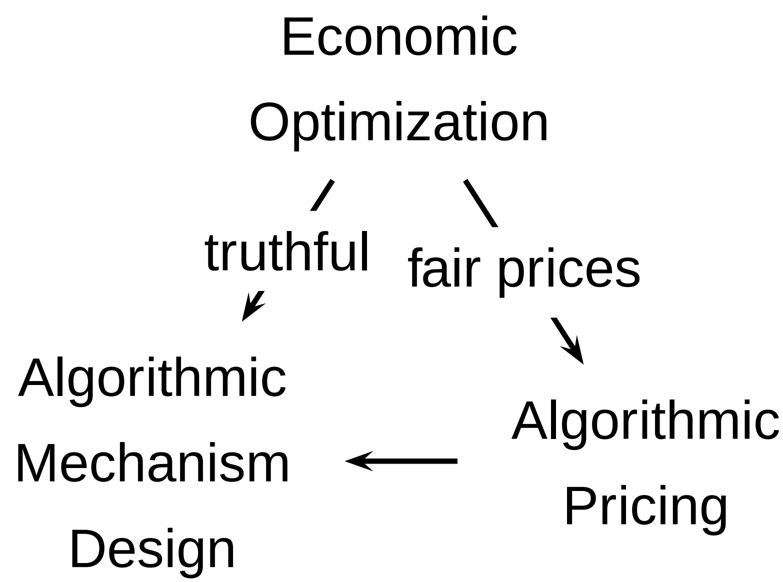

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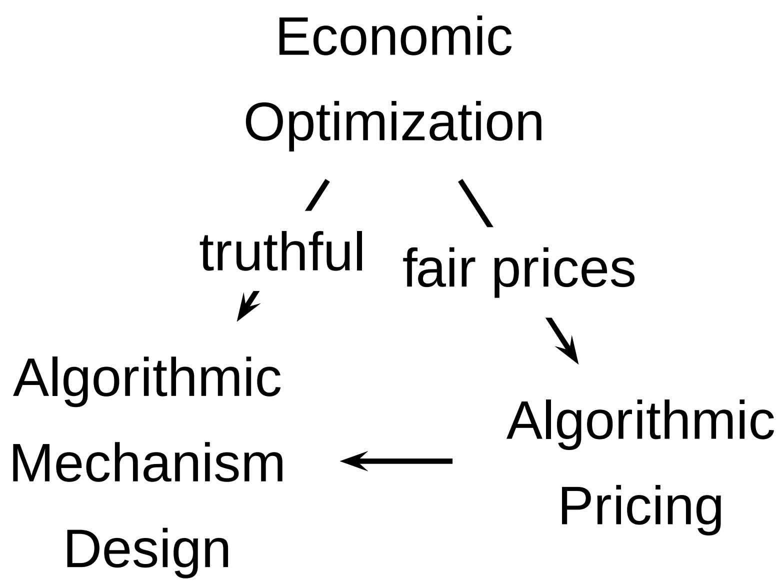

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Algorithmic Mechanism Design ← Algorithmic Pricing

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- **Open:** non-linear objectives
  (e.g., makespan or non-additive costs).